JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 28, 59-76 (1969)

Regular Singular Points in the Nonanalytic Case; Singular Functionals

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Submitted by R. J. Duffin

1. PRELIMINARY REMARKS

The classical theory of the regular singular point for linear differential equations of second order is embedded in the theory of functions of a complex variable. In Section 2, employing fundamental results due to Hille [1], we establish analogs of some of the classical results for the case when the coefficients of the differential equation are of class C' neighboring the singularity. In Section 3 results of Morse and Leighton [2] on principal and nonprincipal solutions of a general second-order linear equation are interpreted for the present type of equation. The foregoing results, in turn, enable us to state theorems concerning the minimum of a singular quadratic functional when its Euler-Jacobi equation possesses a regular singular point in terms of the exponents associated with the singular point. This is done in Section 4. In Section 5 there is a discussion of upper bounds of the first conjugate point for second-order linear differential equations possessing a singular point, and in Section 6 an alternate method is developed that is usually computationally simpler.

2. REGULAR SINGULAR POINTS

If the coefficients $p_1(x)$ and $p_2(x)$ in the differential equation,

$$x^2y'' + xp_1(x)\,y' + p_2(x)\,y = 0 \tag{2.1}$$

are analytic in a neighborhood of the origin, and if the indicial equation

$$\rho^2 - [1 - p_1(0)]\,\rho + p_2(0) = 0 \tag{2.2}$$

* This will acknowledge the partial support of the author by the U.S. Army Research Office (Durham) under Grant numbered DA AROD 31 124 G979.

associated with the point $x = 0$ has real roots σ and τ $(\sigma \geqslant \tau)$, it is well known that the differential equation possesses a solution $x^\sigma z(x)$, where $z(x)$ is analytic near $x = 0$ and $z(0) \neq 0$. In this section we extend this result to the case when $p_1(x)$ and $p_2(x)$ are of class C' on a closed interval containing the origin.

Before stating the principal result of this section we shall transform equation (2.1) in which we now assume that $p_1(x)$ and $p_2(x)$ are of class C' on a closed interval $[0, b]$ $(b > 0)$. We first put the equation in self-adjoint form

$$[r(x)\, y']' + p(x)\, y = 0, \tag{2.3}$$

where

$$r(x) = \exp\left[\int_x^b \frac{p_1(x)}{x}\, dx\right], \qquad p(x) = \frac{p_2(x)}{x^2}\, r(x) \qquad (0 < x \leqslant b).$$

Next, if we set

$$Q(x) = \sigma^2 - [1 - p_1(x)]\, \sigma + p_2(x) \tag{2.4}$$

and substitute $y = x^\sigma z$ in (2.1) or (2.3), we obtain

$$(x^{2\sigma} e^I z')' + x^{2\sigma - 2} e^I Q(x)\, z = 0, \tag{2.5}$$

where

$$I = \int_x^b \frac{p_1(x)}{x}\, dx \qquad (0 < x \leqslant b).$$

In the analytic case, $Q(x)$ could vanish infinitely often in every neighborhood of the origin only if $Q(x) \equiv 0$. We shall require an analogous assumption here—namely, that $Q(x)$ is of one sign near $x = 0$. (The case when $Q(x) \equiv 0$ neighboring the origin is, of course, trivially easy to manage.)

THEOREM 2.1. *If in Eq.* (2.1) *the functions $p_1(x)$ and $p_2(x)$ are of class C' on an interval* $[0, b]$ $(b > 0)$, *if the roots $\sigma \geqslant \tau$ of Eq.* (2.2) *are real, and if $Q(x)$ is of one sign on an interval* $(0, \epsilon)$ $(\epsilon > 0)$, *Eq.* (2.1) *possesses a solution $x^\sigma z(x)$, where $z(x)$ is continuous on the interval* $[0, b]$ *and* $z(0) \neq 0$.

To prove the theorem note first that since $Q(x)$ is of class C' and $Q(0) = 0$ we may write $Q(x) = xq(x)$, where $q(x)$ is continuous on $[0, b]$. Equation (2.5) then becomes

$$(x^{2\sigma} e^I z')' + x^{2\sigma - 1} e^I q(x)\, z = 0. \tag{2.6}$$

Next, we require an extension of a result due to Hille [1]. Hille proved that if $p(x)$ is continuous and eventually of one sign on the interval $[1, \infty)$, a necessary and sufficient condition that the differential equation

$$y'' + p(x)\, y = 0$$

have a solution $y(x)$ with the property that $\lim_{x\to\infty} y(x)$ exist $\neq 0$, $\neq \infty$, is that

$$\int_1^\infty xp(x)\,dx < \infty.$$

We note first that Hille's theorem is valid if the interval $[1, \infty)$ is replaced by $[1, c)$ with the requirement that $p(x)$ be of one sign near $x = c$. His proof also holds *mutatis mutandis* if the interval $[1, \infty)$ is replaced by the interval $(0, b]$ $(b > 0)$. The integral condition above becomes, in this case,

$$\int_0^b xp(x)\,dx < \infty.$$

We require the following extension of Hille's theorem.

LEMMA. *If $r(x)$ and $p(x)$ are continuous with $r(x) > 0$ on the interval $(0, b]$ $(b > 0)$ and if $p(x)$ is of one sign near $x = 0$, a necessary and sufficient condition that the differential equation*

$$[r(x)\,z']' + p(x)\,z = 0 \tag{2.7}$$

possess a solution $z(x)$ with the property that $\lim_{x\to 0+} z(x)$ exist $\neq 0$, $\neq \infty$ is that

$$|J| = \left| \int_0^b p(x) \left[\int_x^b \frac{dx}{r(x)}\right] dx \right| < \infty. \tag{2.8}$$

The analogous condition for the interval $[1, \infty)$ is

$$\left| \int_1^\infty p(x) \left[\int_1^x \frac{dx}{r(x)}\right] dx \right| < \infty.$$

We limit ourselves to the proof of the first statement of the lemma.

If Eq. (2.7) is transformed by means of the substitution

$$t = \int_x^b \frac{dx}{r(x)}, \qquad t_0 = \int_0^b \frac{dx}{r(x)},$$

we obtain, if $\dot{z} = dz/dt$,

$$\ddot{z} + p(x)\,r(x)\,z = 0, \tag{2.9}$$

where $p(x)\,r(x)$ is to be regarded as a function of t determined by the transformation.

Hille's condition applied to Eq. (2.9) becomes

$$\int_0^{t_0} tp(x)\,r(x)\,dt < \infty.$$

But this integral, under the above transformation, becomes the integral J, and the proof is complete.

We return to the proof of the theorem and compute the integral in (2.8) for Eq. (2.6). We have

$$J = \int_0^b x^{2\sigma-1} e^I q(x) \left[\int_x^b x^{-2\sigma} e^{-I}\, dx\right] dx.$$

We next make several observations. Since $p_1(x)$ is of class C' we may write

$$p_1(x) = p_1(0) + xP_1(x),$$

where $P_1(x)$ is continuous on $[0, b]$. It follows that

$$e^I = x^{p_1(0)} E(x),$$

where $E(x)$ is positive and of class C' on $[0, b]$. Further, a reference to Eq. (2.2) and to the fact that $\sigma \geqslant \tau$ indicates that

$$\sigma + \tau = 1 - p_1(0),$$
$$2\sigma + p_1(0) - 1 = \sigma - \tau \geqslant 0. \tag{2.10}$$

Accordingly,

$$J = \int_0^b x^{\sigma-\tau} q(x)\, E(x) \left[\int_x^b x^{-2\sigma-p_1(0)} E_1(x)\, dx\right] dx,$$

where $E_1(x) = [E(x)]^{-1}$. Let

$$M = \max_{0 \leqslant x \leqslant b} \{E_1(x),\ |\, q(x) \,| \cdot E(x)\}.$$

Then

$$|\, J \,| \leqslant M^2 \int_0^b x^{\sigma-\tau} \left[\int_x^b x^{-2\sigma-p_1(0)}\, dx\right] dx.$$

There are two cases to consider: $2\sigma + p_1(0) = 1$, $2\sigma + p_1(0) > 1$; that is, $\sigma = \tau$, $\sigma > \tau$. In either case, J converges, and the theorem is proved.

3. Principal and Nonprincipal Solution

Inasmuch as the zeros of a *principal solution* associated with a point $x = x_0$ of the interval $[0, b]$ are the conjugate points of $x = x_0$ it is important to determine a principal solution of Eq. (2.1). It will be recalled [3] that a principal solution $w(x)$ associated with the point $x = 0$ is a solution $\not\equiv 0$ characterized by the fact that

$$\lim_{x\to 0} \frac{w(x)}{u(x)} = 0, \tag{3.1}$$

for every solution $u(x)$ linearly independent of $w(x)$.

THEOREM 3.1. *The solution $x^\sigma z(x)$ of* (2.1) *is a principal solution corresponding to the point $x = 0$.*

When this result is established it will follow, of course, that $z(x)$ is a principal solution of (2.5) associated with the point $x = 0$.

The proof of the theorem is not difficult. Recall that if $v(x)$ is a solution $\not\equiv 0$ of an Eq. (2.3), a linearly independent solution is

$$v(x) \int_x^b \frac{dx}{r(x)\, v^2(x)} \qquad (0 < x \leqslant b). \tag{3.2}$$

Accordingly, the function

$$y_2(x) = x^\sigma z(x) \int_x^b \frac{dx}{x^{2\sigma+p_1(0)} z^2(x)\, E(x)} \qquad (0 < x \leqslant b),$$

where $z^2(x)\, E(x)$ is continuous and positive at $x = 0$, is a solution of (2.1) linearly independent of $x^\sigma z(x)$.

The ratio

$$\frac{x^\sigma z(x)}{y_2(x)} = \left[\int_x^b \frac{dx}{x^{2\sigma+p_1(0)} z^2(x)\, e(x)}\right]^{-1}.$$

But this integral is less than

$$\frac{1}{m} \int_x^b x^{-2\sigma-p_1(0)}\, dx,$$

where m is a positive constant. It follows at once from (2.10) that this integral diverges as $x \to 0$ (one distinguishes two cases $2\sigma + p_1(0) = 1$, $2\sigma + p_1(0) > 1$). Accordingly,

$$\lim_{x\to 0} \frac{x^\sigma z(x)}{y_2(x)} = 0.$$

Further, if $c_2 \neq 0$,

$$\lim_{x\to 0} \frac{x^\sigma z(x)}{c_1 x^\sigma z(x) + c_2 y_2(x)} = 0,$$

and the theorem is proved.

We have proved incidentally that if σ is real, $x = 0$ cannot be its own first conjugate point. Indeed, this result is also an immediate consequence of Theorem 2.1. Observe also that all solutions $z(x)$ of (2.5) that are linearly independent of a principal solution become infinite neighboring the origin and, hence, so do their derivatives.

NONPRINCIPAL SOLUTIONS. Solutions of Eq. (2.1) that are linearly independent of principal solutions will be described as *nonprincipal* solutions. We shall show that these solutions have properties analogous to nonprincipal solutions in the analytic case.

We return to the solution $y_2(x)$ and write

$$y_2(x) = x^\sigma z(x) \int_x^b \frac{dx}{x^{2\sigma + p_1(0)} z^2(x)\, E(x)} .$$

If $2\sigma + p_1(0) = 1$, then $\sigma = \tau$, and the integral is easily seen to be $0\,(\log x)$, and

$$\begin{aligned} y_2(x) &= x^\sigma z(x)\, 0(\log x) \\ &= x^\sigma 0(\log x). \end{aligned}$$

If $2\sigma + p_1(0) > 1$, then $\sigma > \tau$, the integral is $0(x^{-(\sigma-\tau)})$, and

$$\begin{aligned} y_2(x) &= x^\sigma z(x)\, 0(x^{-(\sigma-\tau)}) \\ &= x^\tau 0(1). \end{aligned}$$

IMAGINARY INDICIAL ROOTS. We shall limit ourselves to showing that if the roots of the indicial Eq. (2.2) are the conjugate imaginaries $\alpha \pm i\beta$ ($\beta \neq 0$), $x = 0$ is its own first conjugate point; that is, every solution of (2.1) vanishes infinitely often near the origin. To demonstrate this we observe that if we substitute

$$y = \left(\frac{x}{e^I}\right)^{1/2} w$$

in Eq. (3.1), we obtain the equation

$$[R(x)\, w']' + P(x)\, w = 0, \tag{3.3}$$

where

$$R(x) = x, \qquad P(x) = \frac{1}{x}\left[-\frac{1}{2} x p_1' + p_2 - \left(\frac{1-p_1}{2}\right)^2\right].$$

It will be sufficient to show that [6, p. 190]

$$\int_0^b \frac{dx}{R(x)} = \infty, \qquad \int_0^b P(x)\, dx = \infty, \tag{3.4}$$

since, if a solution $w(x) \not\equiv 0$ of (3.3) vanishes infinitely often near $x = 0$, so will every solution $y(x)$ of (2.1). That the first condition (3.4) holds is immediate. To show that the second condition is satisfied it will be sufficient to show that $\lim_{x \to 0} xP(x) > 0$. But

$$\lim_{x \to 0} xP(x) = p_2(0) - \left[\frac{1 - p_1(0)}{2}\right]^2$$
$$= \beta^2 > 0.$$

4. A "Regular Singular" Functional

Consider the functional

$$J = \int_0^b [r(x)\, y'^2 - p(x)\, y^2]\, dx, \tag{4.1}$$

where $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on the interval $(0, b]$. In the terminology of an earlier paper [3] the functional J is termed a *principal* quadratic functional. Functions $y(x)$ and the corresponding curves $y = y(x)$ are said to be *A-admissible* on $[0, b]$ if [2, p. 253]

1. $y(x)$ is continuous on $[0, b]$ and $y(0) = y(b) = 0$,
2. $y(x)$ is absolutely continuous and $y'^2(x) \in L$ on each closed subinterval of $(0, b]$.

If

$$\liminf_{x \to 0+} \int_x^b [r(x)\, y'^2 - p(x)\, y^2]\, dx \geqslant 0$$

for each A-admissible curve, J will be said to have a ***minimum limit*** on $[0, b]$. The point $x = 0$ is, in general, a singular point of J and of its corresponding Euler-Jacobi equation.

Conjugate points of $x = 0$ will be defined as in [2]. It will be recalled that $x = 0$ may be its own first conjugate point and that in this case J does not possess a minimum limit. When $x = 0$ is not its own first conjugate point, the conjugate points of $x = 0$, when they exist, are the positive zeros of the principal solution $w(x)$ associated with the point $x = 0$.

The following result is proved in [3, pp. 260-1].

Theorem 4.1. *Let $p(x)$ be positive for x near zero. If*

$$\int_0^b p(x)\, dx = +\infty,$$

J does not possess a minimum limit. If this improper integral exists and if there is no point on $[0, b)$ conjugate to $x = 0$, J does possess a minimum limit on $[0, b]$.

We now specialize J by assuming that

$$r(x) = x^{\alpha}h(x) \qquad \textit{and} \qquad p(x) = x^{\beta}k(x),$$

where $h(x)$ is positive and of class C'' on $[0, b]$, and where $k(x)$ is of class C' on $[0, b]$ and positive on some interval $[0, \epsilon]$ $(0 < \epsilon \leqslant b)$. The functional J becomes

$$J_1 = \int_0^b [x^{\alpha}h(x)\, y'^2 - x^{\beta}k(x)\, y^2]\, dx$$

the Euler-Jacobi equation of which is

$$(x^{\alpha}hy')' + x^{\beta}ky = 0, \tag{4.2}$$

or

$$x^2y'' + x\left(x\frac{h'}{h} + \alpha\right)y' + x^{\beta-\alpha+2}\frac{k}{h}y = 0. \tag{4.3}$$

We distinguish two cases:

CASE I. $\beta - \alpha + 2 = 0$;

CASE II. $\beta - \alpha + 2 \geqslant 1$.

In other cases the coefficient of y in (4.3) will not be of class C' neighboring the origin.

The following result is an immediate consequence of Theorem 4.1.

THEOREM 4.2. *A necessary and sufficient condition that J_1 have a minimum limit is that $x = 0$ have no conjugate point on $[0, b)$ and that $\beta > -1$.*

We shall now attempt to characterize the existence of a minimum limit in terms of the indicial roots of the characteristic equation associated with (4.3). In Section 3 we showed that a necessary and sufficient condition that $x = 0$ be its own first conjugate point is that the indicial roots associated with the singularity at $x = 0$ be conjugate imaginaries. When $x = 0$ is its own first conjugate point, J does not possess a minimum limit [2]. Suppose then that the indicial roots σ and τ are real, with $\sigma \geqslant \tau$. It is known [3, p. 263] that if all solutions of (4.2) pass through the origin J cannot have a minimum limit. It follows that if J_1 possesses a minimum limit, $\tau \leqslant 0$ [cf. 2, p. 273]. Further, if (4.2) has a solution that becomes infinite, as $x \to 0$, and $x = 0$ has no conjugate point on $[0, b)$, J has a minimum limit [3, p. 263]. Accordingly, if $\tau < 0$ and if the conjugate point condition is satisfied, J_1 possesses a minimum limit.

There remains then the case when $\tau = 0$. Two sub-cases must be considered:

(i) $\sigma = \tau = 0$;

(ii) $\tau = 0$, $\sigma = 1 - \alpha > 0$.

When (i) holds, the principal solution $w(x)$ is bounded as $x \to 0$ while

$$\lim_{x \to 0} \frac{w(x)}{u(x)} = 0,$$

for every solution $u(x)$ linearly independent of $w(x)$; consequently, $u(x)$ becomes infinite as $x \to 0$, and J_1 has a minimum limit. On the other hand, (ii) is the ambiguous case. Note that when $\tau = 0$, we must have $\beta - \alpha + 2 \geqslant 1$; that is, we are in Case II above. This includes the case $\alpha = \beta = 0$—that is, the nonsingular case, and J_1 clearly has a minimum limit if the conjugate point condition is satisfied. Case II and Condition (ii) also include the case $\alpha = 0$, $\beta = -1$. In this instance $\beta - \alpha + 2 = 1$, $\tau = 0$, $\sigma = 1 - \alpha = 1$, and because $\beta = -1$, J_1 cannot possess a minimum limit.

We collect these results into the following theorem.

THEOREM 4.3. *When $h(x)$ and $k(x)$ satisfy the conditions given above, the Euler-Jacobi equation of J_1 has a regular singular point at $x = 0$ when either $\beta - \alpha + 2 = 0$ or $\alpha - \beta \leqslant 1$. In the former case, a necessary and sufficient condition that J_1 have a minimum limit on $[0, b]$ is that $x = 0$ have no conjugate point on the interval $[0, b)$ and the smaller indicial root τ associated with the singularity be negative.*

When $\alpha - \beta \leqslant 1$, and the conjugate point condition holds on $[0, b)$, in order that J_1 have a minimum limit it is necessary that $\tau \leqslant 0$, and it is sufficient that either $\tau < 0$ or $\sigma = \tau = 0$. The remaining case $\tau = 0$, $\sigma = 1 - \alpha > 0$ is ambiguous.

Clearly, Theorem 4.3 includes the analytic case—the case when $h(x)$ and $k(x)$ are real analytic functions of the real variable x on the interval $[0, b]$.

Conversely, we observe that if $p_1(x)$ and $p_2(x)$ are of class C' on $[0, b]$, the equation

$$x^2 y'' + x p_1(x) y' + p_2(x) y = 0$$

can be written in self-adjoint form as

$$[P(x) y']' + Q(x) y = 0, \tag{4.4}$$

where

$$P(x) = \exp\left[\int_x^b \frac{p_1(x)}{x}\, dx\right], \qquad Q(x) = \frac{p_2(x)}{x^2} P(x).$$

Inasmuch as $p_1(x)$ and $p_2(x)$ are of class C' we may write

$$P(x) = x^{\alpha}h(x), \qquad Q(x) = x^{\beta}k(x),$$

where $h(x)$ and $k(x)$ are of class C' and $h(x) > 0$ on $[0, b]$, and α and β are suitably chosen real constants. The functional

$$\int_0^b [P(x)\,y'^2 - Q(x)\,y^2]\,dx$$

will be in the form of J_1 and will have (4.4) as its Euler-Jacobi equation. To complete the earlier requirements on $k(x)$ we require that $p_2(x) = x^{\theta}P_2(x)$, where $P_2(x)$ is of one sign on $[0, \epsilon]$ $(0 < \epsilon \leqslant b)$ and θ is a real constant, possibly zero.

It should be noted that if in J_1, $k(x) < 0$ on $[0, \epsilon]$ $(0 < \epsilon \leqslant b)$, the minimum limit of J_1 always exists if the conjugate point condition is satisfied on on $[0, b)$ [*2*, p. 266], as is to be expected.

5. Upper Bounds for the First Conjugate Point

A well-known method for finding an upper bound for the first conjugate point in the nonsingular case is illustrated by the following example. Consider the differential equation

$$y'' + y = 0 \tag{5.1}$$

and the associated quadratic functional

$$J = \int_0^b (y'^2 - y^2)\,dx \qquad (b > 0) \tag{5.2}$$

of which (5.1) is the Euler-Jacobi Equation. It is known [see, e.g., *5*] that if $y(x)$ is any function of class C' on $[0, b]$ for which $y(0) = y(b) = 0$ and $J < 0$, every solution of (5.1) that vanishes at $x = 0$ must vanish again on the open interval $(0, b)$. The function $y(x) = x(b - x)$ gives the value

$$\frac{b^3}{30}(10 - b^2)$$

to J; accordingly, the first positive conjugate point $x = c$ of $x = 0$ has the property that $c < \sqrt{10} = 3.16$ (approximately). The conjugate point occurs, of course, at $x = \pi$. More precisely, one may argue that $c < \sqrt{10}$ provided $x(b - x)$ is not a solution of (5.1); otherwise, c would be precisely equal to $\sqrt{10}$.

Additional care is needed when the differential equation has a singular point, as the following example indicates. It is not difficult to show that $xJ_0(x)$ is a solution of the (modified) Bessel differential equation

$$x^2y'' - xy' + (1 + x^2)y = 0 \tag{5.3}$$

and that it is a principal solution associated with the regular singular point at $x = 0$. The first positive zero of $xJ_0(x)$ is approximately 2.40.

To attempt to follow the method of the previous example to determine an upper bound for this zero we would write (5.3) in self-adjoint form

$$\left(\frac{1}{x}y'\right)' + \left(\frac{1}{x} + \frac{1}{x^3}\right)y = 0, \tag*{(5.3)$'$}$$

and consider the corresponding quadratic functional

$$J = \int_0^b \left[\frac{1}{x}y'^2 - \left(\frac{1}{x} + \frac{1}{x^3}\right)y^2\right] dx. \tag{5.4}$$

Here,

$$p(x) = \frac{1}{x} + \frac{1}{x^3}$$

and

$$\int_0 p(x)\, dx = +\infty;$$

consequently, J does not possess a minimum limit. Accordingly, the method of the previous example does not apply.

We note that the fact that J does not possess a minimum limit can be seen directly, for if we evaluate J along the curve $y = x(b - x)$, we have

$$J = -\frac{b^2}{12}(b^2 + 6). \tag{5.5}$$

The right-hand member of (5.5) is negative for all values of $b \neq 0$.

Considerations such as these lead us to the following ideas. Consider the functional

$$J = \int_0^b [r(x)\,y'^2 - p(x)\,y^2]\, dx, \tag{5.6}$$

where $r(x)$ and $p(x)$ are continuous with $r(x) > 0$ on the interval $(0, b]$. Suppose further that $p(x)$ is of one sign near $x = 0$. Recall that if there is no conjugate point of $x = 0$ on the interval $[0, b)$ and if $p(x) > 0$ near $x = 0$, a necessary and sufficient condition that J possess a minimum limit is that

$$\int_0 p(x)\,dx < \infty.$$

The Euler-Jacobi equation associated with (5.6) is, of course,

$$[r(x)\,y']' + p(x)\,y = 0. \tag{5.7}$$

Let $u(x)$ be a function that is positive and of class C' on the interval $(0, b]$ and such that either

$$P(x) = u(x)\,[(r(x)\,u'(x))' + p(x)\,u(x)] \tag{5.8}$$

is $\leqslant 0$ near $x = 0$, or $P(x) > 0$ near $x = 0$ and

$$\int_0 P(x)\,dx < \infty. \tag{5.9}$$

We transform Eq. (5.7) by means of the substitution

$$y = u(x)\,z, \tag{5.10}$$

and we have

$$[R(x)\,z']' + P(x)\,z = 0, \tag{5.11}$$

where $P(x)$ is given by (5.8) and

$$R(x) = r(x)\,u^2(x).$$

Observe that the positive zeros of a solution $y(x)$ of (5.7) are precisely the positive zeros of the corresponding solution $z(x)$ of (5.11), and conversely. "Corresponding" solutions $y(x)$ and $z(x)$ are those determined by (5.10). Further, it follows at once from (5.10) that if $w(x)$ is a principal solution of (5.7) associated with the point $x = 0$, $w(x)/u(x)$ is a principal solution of (5.11) associated with $x = 0$.

The functional

$$J_1 = \int_0^b [R(x)\,z'^2 - P(x)\,z^2]\,dx \tag{5.12}$$

has Eq. (5.11) as its Euler-Jacobi Equation, and since either $P(x) \leqslant 0$ near $x = 0$ or $P(x) > 0$ near $x = 0$ and (5.9) holds, J_1 has a minimum limit among A-admissible functions $z(x)$ if and only if the point $x = 0$ has no conjugate point on the interval $[0, b)$. We can, accordingly, employ A-admissible functions $z(x)$ in J_1 to compute upper bounds for the first conjugate point of $x = 0$ for the functional J_1, and hence for the first conjugate point of $x = 0$ for the functional J in (5.6).

Let us apply these ideas to Eq. (5.3)'. If we set $u(x) = x^m$ we have

$$R(x) = x^{2m-1}, \qquad P(x) = (m-1)^2\, x^{2m-3} + x^{2m-1}.$$

$P(x)$ is positive near $x = 0$, and many convenient choices of m are available that insure that

$$\int_0 P(x)\, dx < \infty.$$

We choose $m = 1$, and have $R(x) = P(x) = x$, and

$$J_1 = \int_0^b x(z'^2 - z^2)\, dx.$$

If next we compute J_1 along the curve $z = x(b - x)$, we have

$$J_1 = \frac{b^4}{60}(10 - b^2).$$

That is, the first conjugate point $x = c$ of $x = 0$ satisfies the condition $c < \sqrt{10}$.

A better bound can be obtained by the substitution $z = x^\alpha(b - x)$ $(0 < \alpha < 1)$. In this case,

$$J_1 = b^{2\alpha+2}\left[\frac{1}{2(2\alpha+1)} - \frac{2b^2}{(2\alpha+2)(2\alpha+3)(2\alpha+4)}\right]. \tag{5.13}$$

Accordingly,

$$c^2 < \frac{(\alpha+1)(\alpha+2)(2\alpha+3)}{2\alpha+1}, \tag{5.14}$$

for each $\alpha > 0$. The minimum of the right-hand member for $\alpha \geqslant 0$ occurs when $\alpha = 0$; accordingly,

$$c \leqslant \sqrt{6} = 2.45 \text{ (approx.)}$$

The actual value of c, as noted above, is approximately 2.40.

THE EXISTENCE OF A FUNCTION $u(x)$. We assume that there is a first (positive) conjugate point $x = c$ of $x = 0$ associated with (5.6) and (5.7). It follows [see *2*] that there exists a solution $w(x)$ of (5.7) with the property that $w(c) = 0$ and $w(x) > 0$ $(0 < x < c)$. We seek to determine an upper limit b for J, where $b > c$. If, then, we define

$$\begin{aligned} u(x) &= w(x) \qquad (0 < x \leqslant c - \epsilon), \\ &= v(x) \qquad (c - \epsilon \leqslant x \leqslant b), \end{aligned}$$

where $\epsilon > 0$ is small and $v(x)$ is any function of class C' on its interval of definition with the properties

$$v(c - \epsilon) = w(c - \epsilon), \qquad v'(c - \epsilon) = w'(c - \epsilon)$$
$$v(x) > 0 \qquad (c - \epsilon \leqslant x \leqslant b),$$

the resulting function $u(x)$ will have the property that

$$\lim_{x \to 0^+} \int_x^b u(x) \, [(r(x) \, u'(x))' + p(x) \, u(x)] \, dx$$

is finite. It is therefore clear that a function $u(x)$ of the required type always exists. It is clear also, of course, that one ordinarily will wish to choose a more readily accessible $u(x)$. For many important examples a simpler choice, often a power of x, exists.

In the case of a regular singular point at $x = 0$ we have a functional

$$J = \int_0^b [x^\alpha h(x) \, y'^2 - x^{\alpha-2} k(x) \, y^2] \, dx,$$

where, for simplicity, we suppose that $h(x)$, $h'(x)$, $k(x)$ are continuous on the closed interval $[0, b]$ with $h(0) \neq 0$. The Euler-Jacobi Equation for J,

$$x^2 h y'' + x(\alpha h + x h') \, y' + k y = 0,$$

has the indicial equation

$$\rho^2 - (1 - \alpha) \, \rho + \frac{k(0)}{h(0)} = 0.$$

We assume that the indicial roots σ and τ are real with $\sigma \geqslant \tau$.

If we set $u = x^\sigma$, we have

$$\int_0 P(x) \, dx = \int_0 x^{\alpha+2\sigma-2} [\sigma(\alpha + \sigma - 1) \, h + \sigma x h' + k] \, dx$$

$$= \int_0 x^{\alpha+2\sigma-2} P_0(x) \, dx. \tag{5.15}$$

Note that $\sigma + \tau = 1 - \alpha$, and hence that $\alpha + 2\sigma - 1 > 0$, if $\sigma > \tau$. Accordingly, the integral in (5.15) converges when $\sigma > \tau$. It is easy to verify that

$$\lim_{x \to 0^+} P_0(x) = 0.$$

If then we assume further that $h'(x)$ and $k(x)$ are of class C' on $[0, b]$, we have

$$\int_0 P(x) \, dx = \int_0 x^{\alpha+2\sigma-1} P_1(x) \, dx,$$

where $P_1(x)$ is continuous on $[0, b]$, and the integral in (5.15) will converge when $\sigma \geqslant \tau$. Thus, in general, $u(x)$ may be taken as x^σ in the case of a regular singular point at $x = 0$.

6. An Alternative Method; The Rayleigh Quotient

The comments in Section 5 on upper bounds of the smallest positive conjugate point, in the presence of singularities, apply, of course, to computing bounds for the smallest eigenvalue for problems of the type

$$(ry')' + \lambda py = 0, \qquad y(0) = y(1) = 0, \tag{6.1}$$

by use of the Rayleigh quotient

$$\frac{\int_0^b ry'^2\,dx}{\int_0^b py^2\,dx}. \tag{6.2}$$

The principal purpose of this section is to provide a result that will serve as a basis for simpler computation, in general, than that given in Section 5. It is a generalization of a special result due to Marston Morse and the present writer [*2*, p. 276].

To that end, consider the functional

$$J = \int_0^b (ry'^2 - py^2)\,dx \tag{6.3}$$

and its associated Euler-Jacobi Equation

$$(ry')' + py = 0, \tag{6.4}$$

where $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on an interval $(0, b]$ with $p(x) > 0$ on an interval $(0, \epsilon)$ $(0 < \epsilon \leqslant b)$. We assume further that

$$\int_0 p(x)\,dx = +\infty. \tag{6.5}$$

It follows then that J can have no minimum limit among A-admissible functions, as we have noted above.

We continue with the following lemma.

Lemma. *If $y(x)$ is A-admissible and if*

$$\lim_{\epsilon\to 0} \int^b ry'^2\,dx < \infty, \tag{6.6}$$

then

$$[y(x) - y(\epsilon)]^2 \leqslant \int_\epsilon^x \frac{dx}{r(x)} \int_\epsilon^x r(x)\, y'^2(x)\, dx. \tag{6.7}$$

To prove the lemma we employ the Schwarz Inequality and observe that

$$\left[y(x)\Big|_\epsilon^x\right]^2 = \left[\int_\epsilon^x y'(x)\, dx\right]^2 = \left[\int_\epsilon^x \frac{1}{\sqrt{r}} \cdot \sqrt{r}y'\right]^2 \leqslant \int_\epsilon^x \frac{dx}{r} \int_\epsilon^x ry'^2\, dx.$$

We are concerned with functionals J for which $x = 0$ is not its own first conjugate point. It follows, then, from (6.5) that

$$\int_0 \frac{dx}{r(x)} < \infty. \tag{6.8}$$

Accordingly, we may allow $\epsilon \to 0^+$ in (6.7) and obtain

$$y^2(x) \leqslant \int_0^x \frac{dx}{r(x)} \int_0^x r(x)\, y'^2(x)\, dx. \tag{6.9}$$

We shall say that a function $y(x)$ is *r-admissible*, if it is A-admissble and if condition (6.6) holds.

To establish a minimum limit for the functional J among functions $y(x)$ that are r-admissible, it is sufficient (see [*2*]) that there be no conjugate of $x = 0$ on $[0, b)$ and that

$$\lim_{x \to 0} r(x) \frac{u'(x)}{u(x)} y^2(x) = 0, \tag{6.10}$$

where $u(x)$ is any solution of (6.4) that is > 0 on some interval $(0, \epsilon)$ $(\epsilon > 0)$. To establish (6.10) we set

$$z = \frac{ru'}{u}$$

and note that

$$z' = -\left(p + \frac{z^2}{r}\right). \tag{6.11}$$

It follows that if $0 < x \leqslant b$

$$z(b) - z(x) = -\int_x^b \left(p + \frac{z^2}{r}\right) dx, \tag{6.12}$$

and, hence, that $z(x) \to +\infty$, as $x \to 0$. Since $z'(x) < 0$, $z'(x) \to -\infty$, as $x \to 0$.

Next, we write

$$\frac{1}{z(x)}\bigg|_{\epsilon}^{x} = \int_{\epsilon}^{x} \frac{-z'}{z^2}\,dx = \int_{\epsilon}^{x} \left(\frac{p}{z^2} + \frac{1}{r}\right) dx \qquad (0 < \epsilon < x).$$

It follows that

$$\frac{1}{z(x)} - \int_0^x \frac{dx}{r} = \int_0^x \frac{p}{z^2}\,dx. \tag{6.13}$$

From (6.13) we have that

$$z(x) \int_0^x \frac{dx}{r} < 1.$$

It now follows from (6.9) that

$$z(x)\,y^2(x) \leqslant \left[z(x) \int_0^x \frac{dx}{r}\right] \int_0^x r y'^2(x)\,dx,$$

when $y(x)$ is r-admissible, and (6.10) is established.

We have then the following result.

THEOREM 6.1. *If in* (6.3) $r(x)$ *and* $p(x)$ *are continuous and* $r(x) > 0$ *on* $(0, b]$ *with* $p(x) > 0$ *on* $(0, \epsilon)$ $(\epsilon > 0)$, *and if*

$$\int_0 p(x)\,dx = +\infty,$$

a necessary and sufficient condition that J *possess a minimum limit among* r*-admissible functions* $y(x)$ *is that the interval* $[0, b)$ *contain no point conjugate to* $x = 0$.

The sufficiency has just been established, and the necessity was proved in [*2*].

To recapitulate, we recall that in [*4*, p. 119] it was shown that if

$$\limsup_{x \to 0^+} \int_x^b p(x)\,dx < \infty, \tag{6.14}$$

a necessary and sufficient condition that J be afforded an A-minimum limit is that there be no point on $[0, b)$ conjugate to $x = 0$. It was not required that $p(x)$ be of one sign near $x = 0$, in this case. In the present paper, when $p(x) > 0$ near $x = 0$ and the condition

$$\int_0 p(x)\,dx = +\infty. \tag{6.15}$$

holds, we have shown that a similar statement is valid for curves $y = y(x)$ that are r-admissible.

To calculate bounds for the first conjugate point (or for the smallest eigenvalue in a problem of the type (6.1)) the critical step is then to ascertain that $x = 0$ is not its own first conjugate point. Numerous nonoscillation theorems are available for this purpose. After nonoscillation near $x = 0$ is established one may employ A-admissible functions in the usual way provided (6.14) holds. If, on the contrary $p(x) > 0$ near $x = 0$ and (6.15) holds, one may employ r-admissible functions.

It should be pointed out, perhaps, that the results of this section, as well as those in Section 5, are valid ***mutatis mutandis*** on the infinite interval $[0, \infty)$, when the singularity occurs at $x = \infty$.

Note Added in proof. While this paper was in the hands of the printer, the writer stumbled on the paper by M. Bôcher entitled "On regular singular points of linear differential equations of the second order whose coefficients are not necessarily analytic," *Trans. Am. Math. Soc.* **1** (1900), 40–52. Bôcher's paper treats the problem discussed in Sec. 2 of the present paper. The methods of treatment are quite different, as is to be anticipated. The hypotheses are similar, but there are also significant differences. Neither set includes the other.

Perhaps one should add that Bôcher's writing is, as always, a model of clarity.

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